

Non-relativistic Lee Model in Two Dimensional Riemannian Manifolds

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Abstract

This work is a continuation of our previous work (JMP, **48**, 12, pp. 122103-1-122103-20, 2007), where we constructed the non-relativistic Lee model in three dimensional Riemannian manifolds. Here we renormalize the two dimensional version by using the same methods and the results are shortly given since the calculations are basically the same as in the three dimensional model. We also show that the ground state energy is bounded from below due to the upper bound of the heat kernel for compact and Cartan-Hadamard manifolds. In contrast to the construction of the model and the proof of the lower bound of the ground state energy, the mean field approximation to the two dimensional model is not similar to the one in three dimensions and it requires a deeper analysis, which is the main result of this paper.

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1 Introduction

The Lee model was originally introduced in [1] as an exactly soluble (in principle eigenstates and eigenvalues can be exactly found) and a renormalizable model that describes the interaction between a relativistic neutral bosonic field “pions” and two neutral fermionic fields “nucleons”. It is assumed that the nucleon can exist in two different intrinsic states. The particle corresponding to the Bose field is called θ and the particles corresponding to the intrinsic states of the nucleon are called V and N particles. The fermionic field corresponding to the V and N particles are assumed to be spinless for simplicity. Only allowable process is given by

$$V \rightleftharpoons N + \theta \quad (1)$$

and the following process is not allowed

$$N \rightleftharpoons V + \theta, \quad (2)$$

which makes the model rather simple. Although this model is not realistic, the important features of nucleon-pion system can be understood in a relatively simple way and one can get

rid of the infinities without applying perturbation theory techniques. Moreover, the complete non-relativistic version of this model that describes one heavy particle sitting at some fixed point interacting with a field of non-relativistic bosons is as important as its relativistic counterpart. It is much simpler than its relativistic version because only an additive renormalization of the mass difference of the fermions is necessary. It has been studied in a textbook by Henley and Thirring for small number of bosons from the point of view of scattering matrix [2] and there are various other approaches to the model [3, 4, 5, 6, 7, 8, 9]. It is possible to look at the same problem from the point of view of the resolvent of the Hamiltonian in a Fock space formalism with arbitrary number of bosons (in fact there is a conserved quantity which allows us to restrict the problem to the direct sum of n and $n + 1$ boson sectors). This is achieved in a very interesting unpublished paper by S. G. Rajeev [10], in which a new non-perturbative formulation of renormalization for some models with contact interactions has been proposed.

This paper is a natural continuation of our previous work [11] and we extend the three dimensional model constructed there to the two dimensional one. In [11], we discussed the non-relativistic Lee model on three dimensional Riemannian manifolds by following [10] and renormalized the model by the help of heat kernel techniques with the hope that one may understand the nature of renormalization on general curved spaces better. In fact, the idea developed in [10] has also been applied to point interactions in [12, 13, 14] and to the relativistic Lee model in [15]. In this paper, we are not going to review the ideas developed in [10] and [11]. Instead, we recommend the reader to read through these papers. The construction of the model is exactly the same as the one in three dimensions. It is again based on finding the resolvent of the regularized Hamiltonian H_ϵ and showing that a well-definite finite limit of the resolvent exists as $\epsilon \rightarrow 0^+$ (called renormalization) with the help of heat kernel. We then prove that the ground state energy for a fixed number of bosons is bounded from below, using the upper bound estimates of heat kernel for some classes of Riemannian manifolds. Finally, we study the model in the mean field approximation for compact and non-compact manifolds separately. Although the construction of the model and lower bound of the ground state energy are based on the same calculations as in three dimensions, *the mean field approximation in two dimensions requires a deeper analysis* as we shall see, which is the main result of the present work.

The paper is organized as follows. In the first part, we give a short construction of the model and show that the renormalization can be accomplished on two dimensional Riemannian manifolds. Then, we prove that there exists a lower bound on the ground state energy. Finally, the model is examined in the mean field approximation. In appendix, we prove an inequality which we use in the mean field approximation.

2 Construction of the Model in Two Dimensions

In this section, we give a brief summary of the construction of the model in two dimensions since it has been basically done in [11] for the three dimensional model. We start with the regularized Hamiltonian of the non-relativistic Lee model on two dimensional Riemannian manifold (\mathcal{M}, g) with a cut-off ϵ . In natural units ($\hbar = c = 1$), the regularized Hamiltonian on the local coordinates $x = (x_1, x_2) \in \mathcal{M}$ is

$$H^\epsilon = H_0 + H_{I,\epsilon} , \quad (3)$$

where

$$H_0 = \int_{\mathcal{M}} d^2x \phi_g^\dagger(x) \left(-\frac{1}{2m} \nabla_g^2 + m \right) \phi_g(x) , \quad (4)$$

$$H_{I,\epsilon} = \mu(\epsilon) \left(\frac{1 - \sigma_3}{2} \right) + \lambda \int_{\mathcal{M}} d^2x K_\epsilon(x, a; g) (\phi_g(x) \sigma_- + \phi_g^\dagger(x) \sigma_+) . \quad (5)$$

Here $\phi_g^\dagger(x)$, $\phi_g(x)$ is the bosonic creation-annihilation operators defined on the manifold with the metric structure g and λ is the coupling constant, and x, y refers to points on the manifold \mathcal{M} . Also, $K_\epsilon(x, a; g)$ is the heat kernel on a Riemannian manifold with metric structure g and it converges to the Dirac delta function $\delta_g(x, a)$ around the point a on \mathcal{M} as we take the limit $\epsilon \rightarrow 0^+$. For simplicity, we have changed the notation for the heat kernel to $K_s(x, y; g)$ instead of writing $K_{s/2m}(x, y; g)$ which was used in [11]. We also assume stochastic completeness, that is

$$\int_{\mathcal{M}} d^2x K_s(x, y; g) = 1 . \quad (6)$$

Similar to the flat case, $\mu(\epsilon)$ is defined as a bare mass difference between the V particle (neutron) and the N particle (proton). Although the number of bosons is not conserved in the model, one can derive from the equations of motion that there exists a conserved quantity

$$Q = - \left(\frac{1 + \sigma_3}{2} \right) + \int_{\mathcal{M}} d^2x \phi_g^\dagger(x) \phi_g(x) .$$

Therefore, we can express the regularized Hamiltonian as a 2×2 block split according to \mathbb{C}^2 :

$$H^\epsilon - E = \begin{pmatrix} H_0 - E & \lambda \int_{\mathcal{M}} d^2x K_\epsilon(x, a; g) \phi_g^\dagger(x) \\ \lambda \int_{\mathcal{M}} d^2x K_\epsilon(x, a; g) \phi_g(x) & H_0 - E + \mu(\epsilon) \end{pmatrix} . \quad (7)$$

Then, the regularized resolvent of this Hamiltonian in two dimensions is

$$R^\epsilon(E) = \frac{1}{H^\epsilon - E} = \begin{pmatrix} \alpha_\epsilon & \beta_\epsilon^\dagger \\ \beta_\epsilon & \delta_\epsilon \end{pmatrix} = \begin{pmatrix} a_\epsilon & b_\epsilon^\dagger \\ b_\epsilon & d_\epsilon \end{pmatrix}^{-1} , \quad (8)$$

where

$$\begin{aligned} \alpha_\epsilon &= \frac{1}{H_0 - E} + \frac{1}{H_0 - E} b_\epsilon^\dagger \Phi_\epsilon^{-1}(E) b_\epsilon \frac{1}{H_0 - E} \\ \beta_\epsilon &= -\Phi_\epsilon^{-1}(E) b_\epsilon \frac{1}{H_0 - E} \\ \delta_\epsilon &= \Phi_\epsilon^{-1}(E) \\ b_\epsilon &= \lambda \int_{\mathcal{M}} d^2x K_\epsilon(x, a; g) \phi_g(x) . \end{aligned} \quad (9)$$

Here E should be considered as a complex variable. Most importantly, the operator $\Phi_\epsilon(E)$, called principal operator, is given as

$$\Phi_\epsilon(E) = H_0 - E + \mu(\epsilon) - \lambda^2 \int_{\mathcal{M}^2} d^2x d^2y K_\epsilon(x, a; g) K_\epsilon(y, a; g) \phi_g(x) \frac{1}{H_0 - E} \phi_g^\dagger(y) . \quad (10)$$

After performing normal ordering of this operator, we get

$$\Phi_\epsilon(E) = H_0 - E - \lambda^2 \int_{\epsilon/2}^\infty ds \int_{\mathcal{M}^2} d^2x d^2y K_s(x, a; g) K_s(y, a; g)$$

$$\times \phi_g^\dagger(x) e^{-(s-\epsilon/2)(H_0+2m-E)} \phi_g(y) + \mu + \lambda^2 \int_\epsilon^\infty ds K_s(a, a; g) [e^{-s(m-\mu)} - e^{-(s-\epsilon)(H_0+m-E)}] . \quad (11)$$

If we choose $\mu(\epsilon)$ as

$$\mu(\epsilon) = \mu + \lambda^2 \int_\epsilon^\infty ds K_s(a, a; g) e^{-s(m-\mu)} , \quad (12)$$

where μ is defined as the physical energy of the composite state which consists of a boson and the attractive heavy neutron at the center a , and take the limit $\epsilon \rightarrow 0^+$, we obtain

$$\begin{aligned} \Phi(E) &= H_0 - E + \mu + \lambda^2 \int_0^\infty ds K_s(a, a; g) [e^{-s(m-\mu)} - e^{-s(H_0+m-E)}] \\ &- \lambda^2 \int_0^\infty ds \int_{\mathcal{M}^2} d_g^2 x d_g^2 y K_s(x, a; g) K_s(y, a; g) \phi_g^\dagger(x) e^{-s(H_0+2m-E)} \phi_g(y) . \end{aligned} \quad (13)$$

This is the renormalized form of the principal operator so that we have a well-defined explicit formula for the resolvent of the Hamiltonian in terms of the inverse of the principal operator $\Phi^{-1}(E)$. As we will elaborate in more detail later on, the bound states arise from the roots of the equation

$$\Phi(E)|\Psi\rangle = 0 , \quad (14)$$

corresponding to the poles in the resolvent. Hence, the principal operator determines the bound state spectrum.

3 A Lower Bound on the Ground State Energy

Following exactly the same method developed for three dimensions [11], we again find the upper bound of the norm of the operator $\tilde{U}'(E)$ as

$$\begin{aligned} ||\tilde{U}'(E)|| &\leq n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty ds s e^{-s(nm+\mu-E)} \int_0^1 du_1 \int_0^1 du_2 \int_0^1 du_3 \frac{\delta(u_1 + u_2 + u_3 - 1)}{(u_1 u_2)^{1/2}} \\ &\times [K_{2s(u_1+u_3)}(a, a; g)]^{1/2} [K_{2s(u_2+u_3)}(a, a; g)]^{1/2} , \end{aligned} \quad (15)$$

where Γ denotes the gamma function. For each class of manifolds, there are different upper bounds on the heat kernel so we will consider them separately. We will first consider Cartan-Hadamard manifolds. The diagonal upper bound of the heat kernel for two dimensional Cartan-Hadamard manifolds is given as [16, 17],

$$K_s(x, x; g) \leq \frac{C}{(s/2m)} , \quad (16)$$

for all $x \in \mathcal{M}$, $s > 0$, and C is a constant. Using (16) and performing u_3 integral in (15), we get

$$\begin{aligned} ||\tilde{U}'(E)|| &\leq n C m \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty ds e^{-s(nm+\mu-E)} \\ &\times \int_0^1 du_1 \frac{1}{u_1^{1/2}} \frac{1}{(1-u_1)^{1/2}} \int_0^{1-u_1} du_2 \frac{1}{u_2^{1/2}} \frac{1}{(1-u_2)^{1/2}} \end{aligned}$$

$$\leq n C m \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty ds e^{-s(nm+\mu-E)} \left[\int_0^1 du \frac{1}{u^{1/2}} \frac{1}{(1-u)^{1/2}} \right]^2. \quad (17)$$

Evaluating the integrals give the following result

$$||\tilde{U}'(E)|| \leq n C m \pi \lambda^2 \Gamma(2) (nm + \mu - E)^{-1}. \quad (18)$$

Then, the inequality above implies a lower bound for the ground state energy

$$E_{gr} \geq nm + \mu - n\tilde{C}\lambda^2 m, \quad (19)$$

where

$$\tilde{C} = C\pi\Gamma(2). \quad (20)$$

The diagonal upper bound of the heat kernel for two dimensional compact manifolds with Ricci curvature bounded from below by $-K \geq 0$ is given by [18, 19],

$$K_s(x, x; g) \leq \frac{1}{V(\mathcal{M})} + A(s/2m)^{-1}, \quad (21)$$

where A depends on the diameter of the manifold, $d(\mathcal{M})$, the lower bound of the Ricci curvature K , and the volume of the manifold $V(\mathcal{M})$. Then we similarly obtain

$$\begin{aligned} ||\tilde{U}'(E)|| \leq n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty ds s e^{-s(nm+\mu-E)} & \left\{ \int_0^1 \frac{du_1}{(u_1)^{1/2}} \left[\frac{1}{V(\mathcal{M})^{1/2}} \right. \right. \\ & \left. \left. + A^{1/2}(s(1-u_1)/m)^{-1/2} \right] \int_0^{1-u_1} \frac{du_2}{(u_2)^{1/2}} \left[\frac{1}{V(\mathcal{M})^{1/2}} + A^{1/2}(s(1-u_2)/m)^{-1/2} \right] \right\}. \end{aligned} \quad (22)$$

One can even simplify the integrals, that is, the upper bound of the u_2 integral is replaced with 1 and the square roots of the sums are replaced with the sums of the square roots at the cost of getting less sharp bound on the norm of $\tilde{U}'(E)$. Integrating with respect to u_1 , u_2 and s , we have

$$||\tilde{U}'(E)|| \leq n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \left[\frac{4}{V(\mathcal{M})} \frac{1}{(nm + \mu - E)^2} + \frac{4A^{1/2}m^{1/2}\pi^{1/2}\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{V(\mathcal{M})^{1/2}(nm + \mu - E)^{\frac{3}{2}}} + \frac{Am\pi\Gamma(\frac{1}{2})^2}{(nm + \mu - E)} \right]. \quad (23)$$

In order to get an explicit solution of this inequality, let us put a further natural assumption $nm + \mu - E > \mu$. Then, we find

$$||\tilde{U}'(E)|| \leq n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \left[\frac{4}{V(\mathcal{M})\mu} + \frac{4A^{1/2}m^{1/2}\pi^{1/2}\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\mu^{\frac{1}{2}}V(\mathcal{M})^{1/2}} + Am\pi\Gamma(1/2)^2 \right] \frac{1}{(nm + \mu - E)}. \quad (24)$$

Now if we impose the strict positivity of the principal operator, we obtain

$$E_{gr} \geq nm + \mu - n\lambda^2 F, \quad (25)$$

where

$$F = \frac{\Gamma(2)}{\Gamma(1/2)^2} \left[\frac{4}{V(\mathcal{M})\mu} + \frac{4A^{1/2}m^{1/2}\pi^{1/2}\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\mu^{\frac{1}{2}}V(\mathcal{M})^{1/2}} + Am\pi\Gamma(1/2)^2 \right]. \quad (26)$$

Therefore, the lower bounds on the ground state energies for different classes of manifolds (19) and (25) are of almost the same form up to a constant factor so the form of the lower bound has a general character. From the general form of the lower bounds, we conclude that for each sector with a fixed number of bosons, there exists a ground state. However, in two dimensions, the ground state energy bound that we have found diverges linearly as the number of bosons increases whereas in the three dimensional case it diverges quadratically. Therefore, unlike the three dimensional problem, these estimates with our present analysis are good enough to prove the existence of the thermodynamic limit in two dimensions. We will now study large n or the thermodynamic limit of the model by a kind of mean field approximation, yet this requires a more delicate analysis than the three dimensional case.

4 Mean Field Approximation

Before applying the mean field approximation, we first clarify one point about this approximation, which has not been mentioned in our previous work. It is well known that the residue of the resolvent at its isolated pole μ is the projection operator \mathbb{P}_μ to the corresponding eigenspace of the Hamiltonian

$$\mathbb{P}_\mu = -\frac{1}{2\pi i} \oint_{\Gamma_\mu} dE R(E) , \quad (27)$$

where Γ_μ is a small contour enclosing the isolated eigenvalue μ in the complex energy plane [20]. For the moment, let us consider only the first diagonal element of the resolvent, $\alpha(E)$ and choose the contour enclosing the ground state energy E_{gr} which is a well-defined point on the real axis thanks to the bound given in the previous section. Then, the above integral of $\alpha(E)$ gives the projection to eigenspace $|\Psi_0\rangle\langle\Psi_0|$ corresponding to the minimum eigenvalue.

From (13), it is easily seen that the principal operator formally satisfies $\Phi^\dagger(E) = \Phi(E^*)$. As will be shown elsewhere [21], $\Phi(E)$ defines a self-adjoint holomorphic family of type A [22], so that we can apply the spectral theorem for the principal operator or inverse of it. Moreover, holomorphic functional calculus applies to this family. Most importantly, it can be shown that this family has a common dense domain which corresponds to (an operator closure of) the domain of H_0 for all sectors.

$$\begin{aligned} \Phi^{-1}(E) &= \sum_k \frac{1}{\omega_k(E)} \mathbb{P}_k(E) + \int_\sigma d\omega(E) \frac{1}{\omega(E)} \mathbb{P}_\omega(E) \\ &= \sum_k \frac{1}{\omega_k(E)} |\omega_k(E)\rangle\langle\omega_k(E)| + \int_{\sigma(\Phi)} d\omega(E) \frac{1}{\omega(E)} |\omega(E)\rangle\langle\omega(E)| , \end{aligned} \quad (28)$$

where $\omega(E)$ (ω_k in the discrete case) and $|\omega(E)\rangle$ ($|\omega_k(E)\rangle$ in the discrete case) are the eigenvalues and the eigenvectors of the principal operator, respectively. We assume that the principal operator has discrete, assumed non-degenerate, as well as continuous eigenvalues and the bottom of the spectrum corresponds to an eigenvalue. The integrals here are taken over the continuous spectrum $\sigma(\Phi)$ of the principal operator (for simplicity, we write it formally, it should be written more precisely as a Riemann-Stieltjes integral). Due to Feynman-Hellman theorem, we have

$$\begin{aligned} \frac{\partial \omega_k}{\partial E} &= \langle\omega_k(E)| \frac{\partial \Phi(E)}{\partial E} |\omega_k(E)\rangle \\ &= - \left(1 + \lambda^2 \int_0^\infty ds \, s \, K_s(a, a; g) \left\| e^{-\frac{s}{2}(H_0 - \mu + E)} |\omega_k(E)\rangle \right\|^2 \right) \end{aligned}$$

$$+\lambda^2 \int_0^\infty ds \left\| e^{-\frac{s}{2}(H_0 - \mu + E)} \int_{\mathcal{M}} d_g^2 x K_s(x, a; g) \phi_g(x) |\omega_k(E)\rangle \right\|^2 < 0, \quad (29)$$

by using the positivity of the heat kernel. Note that the operator valued distributions $\phi_g(x)$ becomes well defined by taking a convolution with the heat kernel in the last term. The bound state spectrum corresponds to the solutions of the zero eigenvalues of the principal operator. $\omega_k(E)$'s flow with E due to (29), so that the ground state corresponds to the zero of the minimum eigenvalue $\omega_0(E)$ of $\Phi(E)$. Let us expand the minimum eigenvalue $\omega_0(E)$ near the bound state energy E_{gr}

$$\omega_0(E) = \omega_0(E_{gr}) + (E - E_{gr}) \frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} + \dots = (E - E_{gr}) \frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} + \dots. \quad (30)$$

Using this result and the residue theorem in (27), we obtain

$$(H_0 - E_{gr})^{-1} \phi^\dagger(a) \left(-\frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} \right)^{-1} |\omega_0(E_{gr})\rangle \langle \omega_0(E_{gr})| \phi(a) (H_0 - E_{gr})^{-1} \quad (31)$$

There is no other pole coming from $(H_0 - E)^{-1}$ near E_{gr} since we assume $E_{gr} < nm$, and no other terms for $k \neq 0$ contribute to the integral around E_{gr} . Let us assume that the ground state eigenvector of the principal operator is

$$|\omega_0(E_{gr})\rangle = \int_{\mathcal{M}^{n-1}} d_g^2 x_1 \cdots d_g^2 x_{n-1} \psi_0(x_1, \dots, x_{n-1}) |x_1 \cdots x_{n-1}\rangle. \quad (32)$$

By using the eigenfunction expansion of the creation and the annihilation operators and their commutation relations, we shall shift all creation operators $\phi_g^\dagger(x)$ to the leftmost

$$\begin{aligned} \frac{1}{H_0 - E} \phi_g^\dagger(a) \phi_g^\dagger(x_1) \cdots \phi_g^\dagger(x_{n-1}) &= \int_{\mathcal{M}^n} d_g^2 y_1 \cdots d_g^2 y_n \phi_g^\dagger(y_1) \cdots \phi_g^\dagger(y_n) \\ &\times \int_0^\infty ds e^{-s(H_0 - E + nm)} K_s(y_1, a; g) K_s(y_2, x_1; g) \cdots K_s(y_n, x_{n-1}; g), \end{aligned} \quad (33)$$

and all annihilation operators $\phi_g(x)$ to the rightmost

$$\begin{aligned} \phi_g(a) \phi_g(x_1) \cdots \phi_g(x_{n-1}) \frac{1}{H_0 - E} &= \int_{\mathcal{M}^n} d_g^2 y_1 \cdots d_g^2 y_n \int_0^\infty ds e^{-s(H_0 - E + nm)} \\ &\times K_s(y_1, a; g) K_s(y_2, x_1; g) \cdots K_s(y_n, x_{n-1}; g) \phi_g(y_1) \cdots \phi_g(y_n), \end{aligned} \quad (34)$$

which are the generalized versions of the equations we first used in [11]. Therefore, from the equation (31), we read the state vector $|\Psi_0\rangle$

$$\begin{aligned} |\Psi_0\rangle &= \int_{\mathcal{M}^n} d_g^2 y_1 \cdots d_g^2 y_n \Psi_0(y_1, \dots, y_n) |y_1 \cdots y_n\rangle \\ &= \int_{\mathcal{M}^n} d_g^2 y_1 \cdots d_g^2 y_n \int_{\mathcal{M}^{n-1}} d_g^2 x_1 \cdots d_g^2 x_{n-1} \frac{1}{n} \sum_{\sigma \in (1 \cdots n)} \int_0^\infty ds e^{-s(nm - E_{gr})} K_s(y_{\sigma(1)}, a; g) \\ &\times K_s(y_{\sigma(2)}, x_1; g) \cdots K_s(y_{\sigma(n)}, x_{n-1}; g) \psi_0(x_1, \dots, x_{n-1}) \left(-\frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} \right)^{-1/2} |y_{\sigma(1)} \cdots y_{\sigma(n)}\rangle \end{aligned} \quad (35)$$

where the sum runs over all cyclic permutations σ of $(123\dots n)$. We will now make a mean field approximation to this model. In standard quantum field theory, one expects that all the bosons have the same wave function $u(x)$ for the limit of large number of bosons $n \rightarrow \infty$ and the wave function of the system has the product form of the one particle wave functions. However, due to the singular structure of our problem, the wave function in (35) can not have a product form in the large n limit. In order to see this, we note that $nm - E_{gr}$ is the crucial factor. If $nm - E_{gr} = O(n^\alpha)$ where α is a positive exponent, we could get a simplification. To demonstrate this, let us define $nm - E_{gr} = (1 - y[v])\chi - 2my[v]$, where $y[v] < 1$. As we will see, χ is what we typically estimate, and the variable $y[v]$ is related to the scaled kinetic energy functional. Indeed, for compact manifolds we will see that $\chi \approx n^{1/2}$ and $y[v] \approx 0$. We scale $s = s'/[(1 - y[v])\chi - 2my[v]]$ and as $n \rightarrow \infty$, all integrals of the heat kernels are peaked around $y_{\sigma(k)}$. (This is clear from the property of the heat kernel that $K_s(x, y; g) \rightarrow \delta_g(x, y)$ in the sense of distributions as $s \rightarrow 0^+$ and also from the stochastic completeness assumption). Then, all integrals of $x_{\sigma(l)}$ are

$$\int_{\mathcal{M}} d_g^2 x_l K_{s/[(1-y)\chi-2my]}(x_l, y_{\sigma(l+1)}) \psi_0(x_1, \dots, x_l, \dots, x_{n-1}) \approx \psi_0(x_1, \dots, y_{\sigma(l+1)}, \dots, x_{n-1}), \quad (36)$$

for $l = 1, \dots, n-1$ as $n \rightarrow \infty$ and state $|\Psi_0\rangle$ becomes

$$|\Psi_0\rangle \approx \int_{\mathcal{M}^n} d_g^2 y_1 \cdots d_g^2 y_n \frac{1}{n} \sum_{\sigma \in (1\dots n)} \int_0^\infty ds e^{-s(nm-E_{gr})} K_s(y_{\sigma(1)}, a; g) \psi_0(y_{\sigma(2)}, \dots, y_{\sigma(n)}) \left(-\frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} \right)^{-1/2} |y_{\sigma(1)} \cdots y_{\sigma(n)}\rangle. \quad (37)$$

It is important to note that $|\Psi_0\rangle$ is not in the domain of H_0 . In order to see this, it is sufficient to consider the following term when we calculate $\langle \Psi_0 | H_0 | \Psi_0 \rangle$

$$\int_{\mathcal{M}} d_g^2 x \int_0^\infty ds_1 e^{-s_1(nm-E_{gr})} K_{s_1}(x, a; g) \left[\int_0^\infty ds_2 e^{-s_2(nm-E_{gr})} \left(-\frac{1}{2m} \right) \nabla_g^2 K_{s_2}(x, a; g) \right] \\ = \int_{\mathcal{M}} d_g^2 x \int_0^\infty ds_1 e^{-s_1(nm-E_{gr})} K_{s_1}(x, a; g) \left[\int_0^\infty ds_2 e^{-s_2(nm-E_{gr})} \left(-\frac{\partial K_{s_2}(x, a; g)}{\partial s_2} \right) \right], \quad (38)$$

where we have used the property that the heat kernel satisfies the heat equation $-\frac{1}{2m} \nabla_g^2 K_s(x, a; g) + \frac{\partial K_s(x, a; g)}{\partial s} = 0$. After applying the integration by parts to the s_2 integral and using the initial condition for the heat kernel $K_s(x, a; g) \rightarrow \delta_g(x, a)$ as $s \rightarrow 0^+$, we find

$$\int_{\mathcal{M}} d_g^2 x \int_0^\infty ds_1 e^{-s_1(nm-E_{gr})} K_{s_1}(x, a; g) \left[\delta_g(x, a) - (nm - E_{gr}) \int_0^\infty ds_2 e^{-s_2(nm-E_{gr})} K_{s_2}(x, a; g) \right] \\ = \int_0^\infty ds_1 e^{-s_1(nm-E_{gr})} K_{s_1}(a, a; g) - (nm - E_{gr}) \int_0^\infty ds_1 e^{-s_1(nm-E_{gr})} \times \int_0^\infty ds_2 e^{-s_2(nm-E_{gr})} K_{s_1+s_2}(a, a; g) \quad (39)$$

where we have used the semi-group property of the heat kernel $\int_{\mathcal{M}} d^2y K_{s_1}(x, y; g) K_{s_2}(y, z; g) = K_{s_1+s_2}(x, z; g)$. After the change of variables $u = s_1 + s_2$ and $v = s_1 - s_2$, we get

$$\int_0^\infty ds_1 e^{-s_1(nm-E_{gr})} K_{s_1}(a, a; g) - (nm - E_{gr}) \int_0^\infty du u e^{-u(nm-E_{gr})} K_u(a, a; g) . \quad (40)$$

The first term is divergent due to the short time asymptotic expansion of the diagonal heat kernel for any Riemannian manifold without boundary [23]

$$K_s(a, a; g) \sim \frac{1}{(4\pi s/2m)} \sum_{k=0}^{\infty} u_k(a, a) (s/2m)^k , \quad (41)$$

where $u_k(a, a,)$ are scalar polynomials in curvature tensor of the manifold and its covariant derivatives at point a . Similar to the problem with point interactions on manifolds which we studied in [13], our problem here can also be considered as a kind of self-adjoint extension since the wave function Ψ_0 does not belong to the domain of the free Hamiltonian. The self-adjoint extension of the free Hamiltonian extends this domain such that the state Ψ_0 is included. Although the wave function Ψ_0 is not in the domain of H_0 , the eigenfunction ψ_0 corresponding to the lowest eigenvalue of $\Phi(E)$ can be taken in the domain of H_0 (see the discussions in [21]).

As a result, $|\Psi_0\rangle$ given in (37) is not in the product form in the large n limit, that is,

$$|\Psi_0\rangle \neq \int_{\mathcal{M}^n} d^2y_1 \cdots d^2y_n \prod_{k=1}^n \Psi_0(y_k) |y_1 \cdots y_n\rangle . \quad (42)$$

The solution takes a kind of convolution of the wave functions in the domain of H_0 with the bound state wave function which is outside of this domain.

Yet, $\Phi(E)$'s lowest eigenfunction may be approximated by a product form for large number of bosons, that is,

$$\psi_0(x_1, \cdots, x_{n-1}) = u(x_1) \cdots u(x_{n-1}) \quad (43)$$

with the normalization

$$||u||^2 = \int_{\mathcal{M}} d^2x |u(x)|^2 = 1 . \quad (44)$$

In our two dimensional problem, for non-compact case, $\chi = O(\ln n)$ as we will see, so the product formula (37) may not be a good approximation. Similarly, in three dimensions for the noncompact case, $\chi = O(1)$. In such a case, $u(x)$'s are not the wave function of bosons when a bound state forms, but related to the correct wave function through (35). In fact, the full wave function of the ground state could be read from

$$\mathbb{P}_0 = \left(\begin{array}{c} |\Psi_0^{(n+1)}\rangle \\ |\omega_0^{(n)}\rangle \end{array} \right) \otimes \left(\begin{array}{c} |\Psi_0^{(n+1)}\rangle \\ |\omega_0^{(n)}\rangle \end{array} \right)^T , \quad (45)$$

where we explicitly denote the boson numbers. We call the eigenvector of the principal operator in the mean field approximation $|u\rangle$ for consistency of notation with our previous paper [11]. In the mean field approximation, the operators are usually approximately replaced by their expectation values in this state i.e., $\langle f(x) \rangle \approx f(\langle x \rangle)$. However, the exact value of the expectation value of an operator is given in terms of cumulant expansion theorem if it converges [24]. Therefore, we assume that the corrections coming from the higher order cumulants are negligibly small and indeed we will see that this assumption is justified for the particular solution

we will find. Therefore, the expectation value of the principal operator by applying the mean field ansatz becomes

$$\begin{aligned} \phi_E[u] &= nh_0[u] - E + \mu + \lambda^2 \int_0^\infty ds K_s(a, a; g) [e^{-s(m-\mu)} - e^{-s(nh_0[u]+m-E)}] \\ &- n\lambda^2 \int_0^\infty ds \int_{\mathcal{M}^2} d_g^2x d_g^2y K_s(x, a; g) K_s(y, a; g) u^*(x) e^{-s(nh_0[u]+2m-E)} u(y) , \end{aligned} \quad (46)$$

and

$$h_0[u] = \int_{\mathcal{M}} d_g^2x \left(\frac{|\nabla_g u(x)|^2}{2m} + m|u(x)|^2 \right) = K[u] + m , \quad (47)$$

where we have taken $(n-1) \approx n$ for $n \gg 1$ and $K[u]$ is called the kinetic energy functional. Now, we must solve the functional equation $\phi_E[u] = 0$ (giving the bound state spectrum of the problem), that is, we solve E as a functional of $u(x)$, and then find the smallest possible value of E with the constraint (44). One can try to write E as a functional of $u(x)$ from the equation $\phi_E[u] = 0$ and apply the variational methods to minimize $E = E[u]$. However, this is a implicit function of E and there is no simple way to solve exactly this functional equation since E is a complicated functional of $u(x)$. Moreover, we have no explicit expression of the heat kernel on any Riemannian manifold to solve E . Nevertheless, it is possible to find a lower bound on the ground state energy without applying the variational calculus techniques.

Since $nh_0[u]$ and E come together in equation (46), it turns out to be convenient to introduce a new variable $\chi = \chi[u]$

$$\chi \equiv nh_0[u] - E . \quad (48)$$

Then, the condition $\phi_E[u] = 0$ gives

$$\begin{aligned} \chi + \mu + \lambda^2 \int_0^\infty ds K_s(a, a; g) [e^{-s(m-\mu)} - e^{-s(\chi+m)}] \\ = n\lambda^2 \int_0^\infty ds \left| \int_{\mathcal{M}} d_g^2x K_s(x, a; g) u(x) \right|^2 e^{-s(\chi+2m)} . \end{aligned} \quad (49)$$

Note that the left hand side is an increasing function of χ while the right hand side is a decreasing function of χ , hence there is a unique solution for χ .

To get a feel for the problem, we consider χ as the dependent variable. We now remove the χ dependence of the right hand side of (49) by first defining a new dimensionless parameter $s' = 2m(2m + \chi)s$ and scaling the metric $\tilde{g}_{ij} = [2m(2m + \chi)]g_{ij}$. Using the scaling property of heat kernel in two dimensions

$$K_s(x, y; g) = \alpha^2 K_{\alpha^2 s}(x, y; \alpha^2 g) , \quad (50)$$

and then defining new dimensionless wave function $v(x)$

$$v(x) \equiv [2m(2m + \chi)]^{-1/2} u(x) , \quad (51)$$

all explicit χ dependence becomes shifted to the left hand side of (49). The condition $\phi_E[u] = 0$ in two dimensions then gives

$$\left(\chi + \mu + \lambda^2 \int_0^\infty ds K_s(a, a; g) [e^{-s(m-\mu)} - e^{-s(\chi+m)}] \right)$$

$$= n\lambda^2(2m) \int_0^\infty ds' \left| \int_{\mathcal{M}} d_{\tilde{g}}^2 x K_{s'}(x, a; \tilde{g}) v(x) \right|^2 e^{-s'/2m} . \quad (52)$$

Of course, $K_{s'}(x, a; \tilde{g})$ has now a dependence on χ but let us assume that by varying $v(x)$ we can get all possible values of the argument. It is important to notice that the left hand side is an increasing function of χ and the right hand side is always positive. Therefore, the left hand side is minimum when $\chi = -\mu$. Let us denote the inverse function of the left hand side as $f_1(nU)$, that is, $\chi = f_1(nU[v])$. Here $U[v]$ denotes the functional on the right hand side except for the factor n . We can express E in terms of the inverse function,

$$E = nm + 2mnK[v] + (nK[v] - 1)f_1(nU) , \quad (53)$$

where $K[v] = \int_{\mathcal{M}} d_{\tilde{g}}^2 x |\nabla_{\tilde{g}} v(x)|^2$ is the dimensionless kinetic energy functional. Hence unless $nK[v] < 1$, the energy is always bigger than $nm + \mu$. As a result, we see that the interesting possibility corresponds to $nK[v] < 1$ case. In the analysis for $nK[v] < 1$, if we follow the same reasoning as in three dimensions, the following integral appears

$$\int_0^\infty ds (1 - e^{-s(\chi+2m)/2m}) K_s(a, a; g) . \quad (54)$$

However, an upper bound to this integral can not be found by using the diagonal upper bounds of the heat kernel given in the previous section because it is divergent for large values of s . Therefore, we must develop a different method to handle the two dimensional problem.

For the case $nK[v] < 1$, we will again consider the problem for compact and noncompact manifolds separately. Using the eigenfunction expansion for the heat kernel [11] and for $v(x) = \sum_{l=0}^\infty v(l) f_l(x; g)$ and taking the integral with respect to s' , we find the right hand side of (52)

$$n(2m)\lambda^2 \sum_{l_1=0}^\infty \sum_{l_2=0}^\infty \frac{1}{1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2}} v^*(l_1) v(l_2) f_{l_1}(a; \tilde{g}) f_{l_2}^*(a; \tilde{g}) , \quad (55)$$

where $f_l(x; \tilde{g})$ and $\bar{\sigma}_l$ are the eigenfunctions and the eigenvalues of the scaled Laplacian $-\nabla_{\tilde{g}}^2$, respectively. Since this is always positive, it is smaller than the following terms by writing the zero modes separately

$$\leq n(2m)\lambda^2 \left(|f_0(a; \tilde{g})|^2 |v(0)|^2 + 2 \left| \sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} \frac{f_0^*(a; \tilde{g}) v^*(0) f_{l_1}(a; \tilde{g}) v(l_1)}{(1 + \bar{\sigma}_{l_1})} \right| + \left| \sum_{l_1 \neq 0} \sum_{l_2 \neq 0} \frac{f_{l_1}^*(a; \tilde{g}) v^*(l_1) f_{l_2}(a; \tilde{g}) v(l_2)}{(1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2})} \right| \right) . \quad (56)$$

Using $f_0(a; \tilde{g}) = \frac{1}{\sqrt{V(\mathcal{M}(\tilde{g}))}}$ and $|v(0)| \leq 1$ we get

$$\leq n(2m)\lambda^2 \left(\frac{1}{V(\mathcal{M}(\tilde{g}))} + \frac{2}{\sqrt{V(\mathcal{M}(\tilde{g}))}} \left| \sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} \frac{f_{l_1}(a; \tilde{g}) v(l_1)}{(1 + \bar{\sigma}_{l_1})} \right| + \left| \sum_{l_1 \neq 0} \sum_{l_2 \neq 0} \frac{f_{l_1}^*(a; \tilde{g}) v^*(l_1) f_{l_2}(a; \tilde{g}) v(l_2)}{(1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2})} \right| \right) . \quad (57)$$

Let us first consider the second term and multiply both numerator and denominator with the factor $\bar{\sigma}_{l_1}^{\frac{1-\epsilon}{2}}$ and then apply Cauchy-Schwartz inequality so that we find

$$\sum_{l_1 \neq 0} \frac{f_{l_1}(a; \tilde{g}) v(l_1)}{(1 + \bar{\sigma}_{l_1})} \leq \left(\sum_{l_1 \neq 0} |v(l_1)|^2 \bar{\sigma}_{l_1}^{1-\epsilon} \right)^{1/2} \left(\sum_{l_1 \neq 0} \frac{|f_{l_1}(a; \tilde{g})|^2}{(1 + \bar{\sigma}_{l_1})^2 \bar{\sigma}_{l_1}^{1-\epsilon}} \right)^{1/2}, \quad (58)$$

where we have chosen $0 < \epsilon < 1/2$. In order to convert the products $(1 + \bar{\sigma}_{l_1})^2 \bar{\sigma}_{l_1}^{1-\epsilon}$ in the denominator into a summation of them, we use a Feynman parametrization

$$\begin{aligned} \frac{1}{(1 + \bar{\sigma}_{l_1})^2 \bar{\sigma}_{l_1}^{1-\epsilon}} &= \frac{\Gamma(3 - \epsilon)}{\Gamma(2)\Gamma(1 - \epsilon)} \int_0^1 du_1 u_1 \int_0^1 du_2 \frac{\delta(u_1 + u_2 - 1) u_2^{-\epsilon}}{(u_1(1 + \bar{\sigma}_{l_1}) + u_2 \bar{\sigma}_{l_1})^{3-\epsilon}} \\ &= \frac{\Gamma(3 - \epsilon)}{\Gamma(2)\Gamma(1 - \epsilon)} \int_0^1 du_1 \frac{u_1(1 - u_1)^{-\epsilon}}{(u_1 + \bar{\sigma}_{l_1})^{3-\epsilon}}. \end{aligned} \quad (59)$$

One can express the factor $\frac{1}{(u_1 + \bar{\sigma}_{l_1})^{3-\epsilon}}$ as an integral of $s^{2-\epsilon} e^{-s(u_1 + \bar{\sigma}_{l_1})}$ due to

$$\frac{1}{a^{k+1}} = \frac{1}{\Gamma(k+1)} \int_0^\infty ds s^k e^{-as}, \quad (60)$$

where $\Re(a) > 0$ and $\Re(k) > -1$. Therefore, equation (59) becomes

$$\frac{1}{(1 + \bar{\sigma}_{l_1})^2 \bar{\sigma}_{l_1}^{1-\epsilon}} = \frac{1}{\Gamma(1 - \epsilon)} \int_0^1 du_1 u_1 (1 - u_1)^{-\epsilon} \int_0^\infty \frac{ds'}{2m} (s'/2m)^{2-\epsilon} e^{-s'(u_1 + \bar{\sigma}_{l_1})/2m}. \quad (61)$$

Using the eigenfunction expansion of the heat kernel, we have

$$\begin{aligned} \sum_{l_1 \neq 0} \frac{|f_{l_1}(a; \tilde{g})|^2}{(1 + \bar{\sigma}_{l_1})^2 \bar{\sigma}_{l_1}^{1-\epsilon}} &= \frac{1}{\Gamma(1 - \epsilon)} \int_0^1 du_1 \frac{1}{(1 - u_1)^\epsilon} \int_0^\infty \frac{ds'}{2m} (s'/2m)^{2-\epsilon} e^{-s'u_1/2m} \\ &\quad \times \left(K_{s'}(a, a; \tilde{g}) - \frac{1}{V(\mathcal{M}(\tilde{g}))} \right). \end{aligned} \quad (62)$$

After we transform the above integral to the original metric g and the time s and take the s integral by using the diagonal upper bound of the heat kernel for compact manifolds (21), we find a upper bound of the above sum

$$\sum_{l_1 \neq 0} \frac{|f_{l_1}(a; \tilde{g})|^2}{(1 + \bar{\sigma}_{l_1})^2 \bar{\sigma}_{l_1}^{1-\epsilon}} \leq \frac{A\Gamma(2 - \epsilon)}{\Gamma(1 - \epsilon)} \int_0^1 du_1 \frac{u_1^{\epsilon-1}}{(1 - u_1)^\epsilon} = \frac{A\pi\Gamma(2 - \epsilon)}{\Gamma(1 - \epsilon) \sin \pi\epsilon}. \quad (63)$$

We now come to the crucial point. The upper bound of the first sum in (58) can be found by using the following inequality (also used in [25]),

$$\bar{\sigma}_1^{1-\epsilon} < \delta + \left(\frac{\epsilon}{\delta} \right)^{\epsilon/1-\epsilon} \bar{\sigma}_1, \quad (64)$$

where $\delta > 0$ and $0 < \epsilon < 1/2$. Note that this inequality applies only to dimensionless variables $\bar{\sigma}_1$, so that is why we use the scaling transformation at the beginning of the problem as opposed

to the three dimensional case. The proof of this inequality is given in Appendix. As a result, for the first sum of (58), we obtain

$$\sum_{l_1 \neq 0} |v(l_1)|^2 \bar{\sigma}_{l_1}^{1-\epsilon} \leq \delta + \left(\frac{\epsilon}{\delta}\right)^{\epsilon/1-\epsilon} K[v], \quad (65)$$

where excluding the zero mode from the sum again gives the kinetic energy functional $K[v]$ since $\bar{\sigma}_0 = 0$. Then, we finally get

$$\begin{aligned} & \left| \frac{2n(2m)\lambda^2}{\sqrt{V(\mathcal{M}(\tilde{g}))}} \sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} \frac{f_{l_1}(a; \tilde{g})v(l_1)}{(1 + \bar{\sigma}_{l_1})} \right| \\ & \leq \frac{2n(2m)\lambda^2}{\sqrt{2m(2m + \chi)V(\mathcal{M}(g))}} \left[\delta + \left(\frac{\epsilon}{\delta}\right)^{\epsilon/1-\epsilon} K[v] \right]^{1/2} \left[\frac{A\pi\Gamma(2 - \epsilon)}{\Gamma(1 - \epsilon) \sin \pi\epsilon} \right]^{1/2}. \end{aligned} \quad (66)$$

Since $\sin \pi\epsilon \geq 2\epsilon$ for $0 \leq \pi\epsilon \leq \pi/2$ (a useful inequality: $\sin \theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$) and $nK[v] < 1$, the last expression is smaller than

$$\frac{2\sqrt{n}\sqrt{2m}\lambda^2}{\sqrt{(2m + \chi)V(\mathcal{M}(g))}} \left[n \left(\frac{\delta}{\epsilon}\right) + \frac{1}{\epsilon} \left(\frac{\epsilon}{\delta}\right)^{\epsilon/1-\epsilon} \right]^{1/2} \left[\frac{A\pi\Gamma(2 - \epsilon)}{2\Gamma(1 - \epsilon)} \right]^{1/2}. \quad (67)$$

By choosing arbitrary constants ϵ and δ , we see that the interaction term brings a contribution of order $O(n)$ to the total energy. However, one can even find a better solution to the large n behavior of the energy. In order to control the energy as $n \rightarrow \infty$, we can assume that the parameters ϵ and δ are sequences in n . Without loss of generality we can assume that $\epsilon(n)$ goes to zero as $n \rightarrow \infty$ (recall that $0 < \epsilon < 1/2$). If we want to find a better large n behavior of the energy, we must choose the sequence $\delta(n)$ such that

$$\frac{n\delta(n)}{\epsilon^2(n)} = O(1). \quad (68)$$

We are now looking for an optimal solution for the energy and tell how fast the sequences $\epsilon(n)$ and $\delta(n)$ must change with n . In order to see this, let us write (67) in the following way

$$\frac{2\sqrt{n}\sqrt{2m}\lambda^2}{\sqrt{(2m + \chi)V(\mathcal{M}(g))}} \left[\frac{n\delta(n)}{\epsilon(n)} + e^{\frac{\epsilon(n)}{1-\epsilon(n)} \ln(\epsilon(n)/\delta(n)) - \ln \epsilon(n)} \right]^{1/2} \left[\frac{A\pi\Gamma(2 - \epsilon(n))}{2\Gamma(1 - \epsilon(n))} \right]^{1/2}. \quad (69)$$

An optimal solution of the sequences can be found in such a way that the exponential term goes asymptotically

$$e^{\frac{\epsilon(n)}{1-\epsilon(n)} \ln(\epsilon(n)/\delta(n)) - \ln \epsilon(n)} = O(\ln n). \quad (70)$$

This implies that we can choose

$$\epsilon(n) = \frac{1}{\ln n}, \quad (71)$$

and as a consequence of (68)

$$\delta(n) = \frac{1}{n \ln^2 n}. \quad (72)$$

Therefore, we obtain the upper bound of (66) for $n \gg 1$

$$\frac{2n(2m)\lambda^2}{\sqrt{V(\mathcal{M}(\tilde{g}))}} \left| \sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} \frac{f_{l_1}(a; \tilde{g})v(l_1)}{(1 + \bar{\sigma}_{l_1})} \right| \leq \frac{\lambda^2 \sqrt{4mA\pi e}}{\sqrt{(2m + \chi)V(\mathcal{M})}} \sqrt{n \ln n}. \quad (73)$$

As for the last term in (57), the procedure outlined above is very similar. However, we now multiply both numerator and denominator of this term with the factor $\bar{\sigma}_{l_1}^{\frac{1-\epsilon}{2}} \bar{\sigma}_{l_2}^{\frac{1-\epsilon}{2}}$ and apply Cauchy-Schwartz inequality and get

$$\begin{aligned} \sum_{l_1 \neq 0} \sum_{l_2 \neq 0} \frac{f_{l_1}^*(a; \tilde{g})v^*(l_1)f_{l_2}(a; \tilde{g})v(l_2)}{(1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2})} \\ \leq \left[\sum_{l_1 \neq 0} |v(l_1)|^2 \bar{\sigma}_{l_1}^{1-\epsilon} \right] \left[\sum_{l_1 \neq 0} \sum_{l_2 \neq 0} \frac{|f_{l_1}(a; \tilde{g})|^2 |f_{l_2}(a; \tilde{g})|^2}{(1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2})^2 \bar{\sigma}_{l_1}^{1-\epsilon} \bar{\sigma}_{l_2}^{1-\epsilon}} \right]^{1/2}. \end{aligned} \quad (74)$$

For simplicity, we can use the following inequality in the second sum $\frac{1}{(1+\bar{\sigma}_{l_1}+\bar{\sigma}_{l_2})^2} \leq \frac{1}{(1+\bar{\sigma}_{l_1})(1+\bar{\sigma}_{l_2})}$ since $\bar{\sigma}_{l_0} = 0$ and obtain an upper bound on (74)

$$\leq \left[\sum_{l_1 \neq 0} |v(l_1)|^2 \bar{\sigma}_{l_1}^{1-\epsilon} \right] \left[\sum_{l_1 \neq 0} \frac{|f_{l_1}(a; \tilde{g})|^2}{(1 + \bar{\sigma}_{l_1}) \bar{\sigma}_{l_1}^{1-\epsilon}} \right]. \quad (75)$$

We again convert the products $(1 + \bar{\sigma}_{l_1}) \bar{\sigma}_{l_1}^{1-\epsilon}$ in the denominator into a summation of them by using a Feynman parametrization

$$\frac{1}{(1 + \bar{\sigma}_{l_1}) \bar{\sigma}_{l_1}^{1-\epsilon}} = \frac{\Gamma(2 - \epsilon)}{\Gamma(1 - \epsilon)} \int_0^1 du_1 \frac{1}{(1 - u_1)^\epsilon (u_1 + \bar{\sigma}_{l_1})^{2-\epsilon}}. \quad (76)$$

Then, we rewrite the factor $\frac{1}{(u_1 + \bar{\sigma}_{l_1})^{2-\epsilon}}$ using (60) and get

$$\begin{aligned} \sum_{l_1 \neq 0} \frac{|f_{l_1}(a; \tilde{g})|^2}{(1 + \bar{\sigma}_{l_1}) \bar{\sigma}_{l_1}^{1-\epsilon}} &= \frac{1}{\Gamma(1 - \epsilon)} \int_0^1 du_1 \frac{1}{(1 - u_1)^\epsilon} \int_0^\infty \frac{ds'}{2m} (s'/2m)^{1-\epsilon} \\ &\quad \times \sum_{l_1 \neq 0} e^{-s'(u_1 + \bar{\sigma}_{l_1})/2m} |f_{l_1}(a; \tilde{g})|^2. \end{aligned} \quad (77)$$

Using the eigenfunction expansion of the heat kernel above leads to

$$\frac{1}{\Gamma(1 - \epsilon)} \int_0^1 du_1 \frac{1}{(1 - u_1)^\epsilon} \int_0^\infty \frac{ds'}{2m} (s'/2m)^{1-\epsilon} e^{-s'u_1/2m} \left(K_{s'}(a, a; \tilde{g}) - \frac{1}{V(\mathcal{M}(\tilde{g}))} \right). \quad (78)$$

Scaling back to the original variables and using the diagonal upper bound of the heat kernel for compact manifolds (21), and taking the s integral, we obtain an upper bound of (77)

$$A \int_0^1 du_1 \frac{u_1^{\epsilon-1}}{(1 - u_1)^\epsilon} = \frac{\pi A}{\sin \pi \epsilon}. \quad (79)$$

The analysis for the first sum in (75) is exactly the same as before. Therefore, the upper bound of the third term in (57) for $n \gg 1$

$$n(2m)\lambda^2 \left| \sum_{\substack{l_1 \neq 0 \\ l_2 \neq 0}} \frac{f_{l_1}^*(a; \tilde{g}) v^*(l_1) f_{l_2}(a; \tilde{g}) v(l_2)}{(1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2})} \right| \leq 2mA\lambda^2 \pi e \ln n . \quad (80)$$

Combining the upper bounds (73) and (80), we finally obtain an upper of (57) for $n \gg 1$

$$\left[\frac{\lambda^2}{(2m + \chi)V(\mathcal{M}(g))} \right] n + \lambda^2 \left[\frac{4m\pi e A}{(2m + \chi)V(\mathcal{M}(g))} \right]^{1/2} n^{1/2} \ln^{1/2} n + \left[2mA\lambda^2 \pi e \right] \ln n . \quad (81)$$

For the left hand side of (52), we have

$$\chi + \mu \leq \chi + \mu + \lambda^2 \int_0^\infty ds K_s(a, a; g) [e^{-s(m-\mu)} - e^{-s(\chi+m)}] , \quad (82)$$

due to the positivity of the heat kernel and $\chi \geq -\mu$. Using this result and the upper bound of the right hand side of (52), we have

$$\chi + \mu \leq \left[\frac{\lambda^2}{(2m + \chi)V(\mathcal{M}(g))} \right] n + \lambda^2 \left[\frac{4m\pi e A}{(2m + \chi)V(\mathcal{M}(g))} \right]^{1/2} n^{1/2} \ln^{1/2} n + \left[2mA\lambda^2 \pi e \right] \ln n . \quad (83)$$

We will solve χ from this inequality for the large values of n . It is important to notice that the right hand side is a monotonically decreasing and the left hand side is a monotonically increasing function of χ . Therefore, if we find a solution to the above equation, say at $\chi = \chi_*$, we conclude that $\chi \leq \chi_*$. It is sufficient to find the leading order term of χ for our purposes so χ_* can be taken as an infinite power series

$$\chi_* \sim a_1 n^{\alpha_1} + a_2 n^{\alpha_2} + \dots \quad (84)$$

with the decreasing power $\alpha_1 > \alpha_2 > \dots$ of n and a_1, a_2, \dots are the coefficients which can be found by substitution. Then we find the leading order term of χ

$$\chi \leq \frac{\lambda}{\sqrt{V(\mathcal{M})}} n^{1/2} \quad (85)$$

for $n \gg 1$. As a result of this, we get the lower bound of the bound state energy E_{gr} for large values of n

$$E_{gr} \sim nm + \mu - \frac{\lambda}{\sqrt{V(\mathcal{M})}} n^{1/2} . \quad (86)$$

As for the noncompact manifolds, the analog expression of (55) is

$$n(2m)\lambda^2 \int \int d\mu(l_1) d\mu(l_2) \frac{1}{1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2}} v^*(l_1) v(l_2) f_{l_1}(a; \tilde{g}) f_{l_2}^*(a; \tilde{g}) , \quad (87)$$

where $d\mu$ here refers to the formal spectral measure. One can think that the sums are replaced with the integrals in the case for noncompact cases and the analysis is basically same as the

one for compact manifolds except that we do not have to bother for extracting the zero mode. Following the same steps for the analog expression of the compact cases, and using the diagonal upper bound of the heat kernel on Cartan-Hadamard manifolds (16), the ground state energy goes like

$$E_{gr} \sim nm + \mu - 2mCe\lambda^2 \ln n. \quad (88)$$

Incidentally, these show that $K[u] \sim \chi K[v] = y[v]\chi n^{-1} \approx y[v]n^{-(1-\alpha)}$ where $\alpha = 1/2$ for compact and $\alpha = 0$ for noncompact manifolds. Hence, the cummulant approximation is expected to give only the term that we kept, and this shows that the approximation is consistent.

The solution actually implies an ansatz for the solution which is possible for compact manifolds. Let us set $u = \frac{1}{\sqrt{V(\mathcal{M})}}$. Then we find for the energy

$$nm - E + \mu + \lambda^2 \int_0^\infty ds K_s(a, a; g) [e^{-s(m-\mu)} - e^{-s(nm-E)}] = n \frac{\lambda^2}{V(\mathcal{M})} \left(\frac{1}{nm - E} \right). \quad (89)$$

Now using a scaling for $s \mapsto s'/(nm - E)$ and using the diagonal asymptotic expansion (41) for $K_s(a, a; g)$,

$$nm - E + \mu + \lambda^2 \int_0^\infty ds' \frac{2m}{4\pi s'} [e^{-s'(m-\mu)/(nm-E)} - e^{-s'}] = n \frac{\lambda^2}{V(\mathcal{M})} \left(\frac{1}{nm - E} \right), \quad (90)$$

we find

$$nm - E + \mu + \frac{m\lambda^2}{2\pi} \ln \left(\frac{nm - E}{m - \mu} \right) = n \frac{\lambda^2}{V(\mathcal{M})} \left(\frac{1}{nm - E} \right). \quad (91)$$

This has the asymptotic solution as claimed in the compact case. Interestingly, the similar ansatz for the three dimensional compact manifolds gives exactly the same type of asymptotics for the energy as in two dimensions, which is better than the upper bound obtained of the mean field analysis (note that one also has $K[u] \sim n^{-1/3}$ so it suggests a constant wave function as an ansatz).

5 Conclusion

In this paper, we considered the non-relativistic Lee model on various class of two dimensional Riemannian manifolds. It is just a continuation of our previous work [11]. The construction of the model and the calculations for lower bound of the ground state energy is almost the same as in three dimensions. However, the mean field approximation in two dimensions is not as simple as in three dimensions and it required more analysis to predict more precisely the large particle limit of the model for compact and non-compact manifolds.

6 Appendix: The Proof of the Inequality

In order to prove the following inequality

$$x^{1-\epsilon} < \delta + \left(\frac{\epsilon}{\delta} \right)^{\frac{\epsilon}{1-\epsilon}} x, \quad (92)$$

where x is a dimensionless variable, let us consider the following polynomial function

$$f(x) = \delta + \left(\frac{\epsilon}{\delta}\right)^{\frac{\epsilon}{1-\epsilon}} x - x^{1-\epsilon} . \quad (93)$$

We assume that $\delta > 0$ and $0 < \epsilon < 1/2$. This function has one extremum point at x_*

$$x_*^\epsilon = (1 - \epsilon) \left(\frac{\delta}{\epsilon}\right)^{\frac{\epsilon}{1-\epsilon}} , \quad (94)$$

and this location x_* corresponds to the minimum. One can easily see that

$$f(x_*) = \delta \left(1 - (1 - \epsilon)^{\frac{1-\epsilon}{\epsilon}}\right) . \quad (95)$$

Since $\epsilon < 1/2$, we have $\frac{1-\epsilon}{\epsilon} > 1$. We can also show that $(1 - \epsilon)^\alpha < 1 - \epsilon$ for $\alpha > 1$ since $1 - \epsilon < 1$ or

$$(1 - \epsilon)^{\frac{1-\epsilon}{\epsilon}} < 1 - \epsilon . \quad (96)$$

Then, we find

$$f(x_*) > \delta\epsilon , \quad (97)$$

and it is always positive. Since this is a global minimum point, we obtain $f(x) > 0$ for all x , which completes the proof.

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